Combining Paraconsistent Logic with Argumentation

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Abstract. One tradition in the logical study of argumentation is to allow for arguments that combine strict and defeasible inference rules, and to derive the strict inference rules from a logic at least as strong as classical logic. An unsolved problem in this tradition is how the trivialising effect of the classical Ex Falso principle can be avoided when two arguments that use defeasible rules have contradictory conclusions. The problem is especially hard since any solution should arguably preserve current results on satisfaction of consistency and logical closure properties. One approach to solve the problem is to replace classical logic as the source for strict rules with a weaker, monotonic paraconsistent logic. This paper explores this approach in the context of the *ASPIC*$^+$ framework for structured argumentation, by instantiating it with a paraconsistent consequence notion of Rescher & Manor (1970). The results are positive: satisfaction of the closure and consistency postulate is proven.

Keywords. *ASPIC*$^+$, strict and defeasible inference, paraconsistent logics, rationality postulates, contamination

1. Introduction

One tradition in the logical study of argumentation is to allow for arguments that combine strict and defeasible inference rules, and to derive the strict inference rules from a logic at least as strong as classical logic. An unsolved problem in this tradition, originally identified by [1,2], is how the trivialising effect of the classical Ex Falso principle can be avoided when two arguments that use defeasible rules have contradictory conclusions. The problem is especially hard since any solution should arguably preserve current results on satisfaction of the consistency and closure postulates of [3].

In a nutshell, the problem is as follows. Suppose two arguments have contradictory conclusions. If the strict inference rules include the Ex Falso principle (that an inconsistent set implies any formula), then these two arguments can be combined into an argument for $\neg\varphi$ for any formula. This combined argument can (depending on the circumstances) defeat any argument for φ. So when there are arguments for contradictory conclusions, any other argument is potentially under threat, which is clearly undesirable. Pollock [1,2] thought that he had avoided this problem by allowing for multiple labellings, but Caminada [4] showed Pollock's solution does not fully avoid contamination. The problem is a genuine one, since there arguably is a real need for argumentation systems that allow for combinations of strict and defeasible inferences and that, moreover, allow for the full reasoning power of a deductive logic. Although for many cases less expressiveness may suffice, a full theory of the logic of argumentation cannot exclude the general case.

Caminada, Carnielli and Dunne [5] formulated a new set of rationality postulates in addition to those of Caminada and Amgoud [3], to characterise cases under which the trivialisation problem is avoided (called the postulates of non-interference and crash-resistance). The problem can then be reformulated as the problem of how to satisfy the new postulates of Caminada, Carnielli and Dunne while preserving known results on satisfying the postulates of Caminada and Amgoud.

To solve the problem, two approaches are possible. One is to change the definitions of the argumentation system, while the other is to derive the strict inference rules from a weaker logic than classical logic. The first approach is taken by Wu [6], who for the *ASPIC*$^+$ framework [7,8] requires that for each argument the set of conclusions of all its subarguments are classically consistent. She shows that this solution works for a restricted version of *ASPIC*$^+$ without preferences, but gives counterexamples to the consistency postulates for the case with preferences.

Another approach to solve the problem is to replace classical logic as the source for strict rules with a weaker, monotonic paraconsistent logic, in order to invalidate the Ex Falso principle as a valid strict inference rule. This paper explores the use of [9]'s paraconsistent consequence notion in the context of the *ASPIC*$^+$ framework for structured argumentation. The choice of *ASPIC*$^+$ for these purposes is justified by its generality and the fact that it captures various other approaches as special cases, such as ABA as studied in [10], various forms of classical argumentation as studied in [11], and various instantiations with Tarskian abstract logics as studied by [12]. Therefore, results in terms of *ASPIC*$^+$ are representative for a large class of argumentation systems.

For simplicity we take it for granted that with any paraconsistent logic, *ASPIC*$^+$ satisfies the postulates of [5] because of the absence of the Ex Falso principle as a strict inference rule. The focus is on the question whether with [9]'s consequence notion *ASPIC*$^+$ still satisfies the consistency and closure postulates.

The rest of this paper is organised as follows. First the *ASPIC*$^+$ framework is summarised and the problem is illustrated in more detail. Then [9]'s paraconsistent consequence notion is presented and embedded in an adapted version of *ASPIC*$^+$. Then the main contribution of this paper is given, in the form of proofs of satisfaction of the closure and consistency postulates (exploiting recent results of [13]).

2. The *ASPIC*$^+$ Framework

In this section, [14]'s theory of abstract argumentation and its use by the *ASPIC*$^+$ framework are briefly reviewed.

An *abstract argumentation framework* (AF) is a pair $(\mathcal{A}, \mathcal{D})$, where $\mathcal{A}$ is a set of *arguments* and $\mathcal{D} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary relation of *defeat*. An argument A defeats argument B if $(A, B) \in \mathcal{D}$. A set S of arguments defeats an argument B if there is an argument $A \in S$ such that A defeats B. A set S defeats a set S' if there is an argument $A \in S'$ such that S defeats A. A set of arguments is said to be *conflict-free* if it does not attack itself; otherwise it is *conflicting*. A set is *admissible* if it is conflict-free and defends itself by attacking each argument attacking S. *Extensions* are admissible sets with some additional properties. They can be defined according to the function $F : 2^{\mathcal{A}} \rightarrow 2^{\mathcal{A}}$ such that $F(S) = \{A | A$ A is defended by $S\}$. A *complete* extension is an admissible set that contains each argument it defends; this is any conflict-free fixed point of F. A *preferred* extension is a maximal conflict-free fixed point of F. A minimal conflict-free fixed point corresponds to a *grounded* extension. A *stable* extension is conflict-free and attacks all arguments not belonging it.

The *ASPIC*$^+$ framework [7,8] gives structure to Dung's arguments and defeat relation. It defines arguments as inference trees formed by applying strict or defeasible inference rules to premises formulated in some logical language. Informally, if an inference rule's antecedents are accepted, then if the rule is strict, its consequent must be accepted *no matter what*, while if the rule is defeasible, its consequent must be accepted *if there are no good reasons not to accept it*. Arguments can be attacked on their (ordinary) premises and on their applications of defeasible inference rules. Some attacks succeed as *defeats*, which is partly determined by preferences. Below the version of *ASPIC*$^+$ defined in [8] is presented, more precisely, the special case with symmetric negation.

ASPIC$^+$ is not a system but a framework for specifying systems. It defines the notion of an abstract *argumentation system* as a structure consisting of a logical language $\mathcal{L}$ closed under negation, a set $\mathcal{R}$ consisting of two subsets $\mathcal{R}_s$ and $\mathcal{R}_d$ of strict and defeasible inference rules, and a naming convention n in $\mathcal{L}$ for defeasible rules in order to talk about the applicability of defeasible rules in $\mathcal{L}$. Thus informally, $n(r)$ is a wff in $\mathcal{L}$, which says that rule $r \in \mathcal{R}$ is applicable.

Definition 1. [**Argumentation systems**] An *argumentation system* is a triple $AS = (\mathcal{L}, \mathcal{R}, n)$ where:

- $\mathcal{L}$ is a logical language closed under negation ($\neg$).
- $\mathcal{R} = \mathcal{R}_s \cup \mathcal{R}_d$ is a set of strict ($\mathcal{R}_s$) and defeasible ($\mathcal{R}_d$) inference rules of the form $\varphi_1, \ldots, \varphi_n \rightarrow \varphi$ and $\varphi_1, \ldots, \varphi_n \Rightarrow \varphi$ respectively (where φ_i, φ are meta-variables ranging over wff in $\mathcal{L}$), and $\mathcal{R}_s \cap \mathcal{R}_d = \emptyset$.
- $n : \mathcal{R}_d \longrightarrow \mathcal{L}$ is a naming convention for defeasible rules.

The following notation is used $\psi = -\varphi$ just in case $\psi = \neg\varphi$ or $\varphi = \neg\psi$.

Definition 2. [**Knowledge bases**] A *knowledge base* in an $AS = (\mathcal{L}, \mathcal{R}, n)$ is a set $\mathcal{K} \subseteq \mathcal{L}$ consisting of two disjoint subsets $\mathcal{K}_n$ and $\mathcal{K}_p$ (the *necessary* and *ordinary premises*).

Intuitively, the necessary premises are certain knowledge and thus cannot be attacked, whereas the ordinary premises are uncertain and thus can be attacked.

Definition 3. [**Consistency and strict closure**] For any $X \subseteq \mathcal{L}$, let the closure of X under strict rules, denoted $Cl(X)$, be the smallest set containing X and the consequent of any strict rule in $\mathcal{R}_s$ whose antecedents are in $Cl(X)$. Then a set $X \subseteq \mathcal{L}$ is

- *directly consistent* iff $\nexists\, \psi, \varphi \in X$ such that $\psi = -\varphi$;
- *indirectly consistent* iff $Cl(X)$ is directly consistent.

Arguments can be constructed step-by-step from knowledge bases by chaining inference rules into trees. In what follows, for a given argument the function `Prem` returns all its premises, $\texttt{Conc}$ returns its conclusion and $\texttt{Sub}$ returns all its sub-arguments, while $\texttt{TopRule}$ returns the last rule used in the argument.

Definition 4. [**Argument**] An *argument* A on the basis of a knowledge base $\mathcal{K}$ in an argumentation system $(\mathcal{L}, \mathcal{R}, n)$ is:

1. φ if $\varphi \in \mathcal{K}$ with: $\texttt{Prem}(A) = \{\varphi\}$; $\texttt{Conc}(A) = \varphi$; $\texttt{Sub}(A) = \{\varphi\}$; $\texttt{TopRule}(A)$ = undefined.
2. $A_1, \ldots A_n \rightarrow/\Rightarrow \psi$ if $A_1, \ldots, A_n$ are arguments such that R_s/R_d contains the strict/defeasible rule $\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow/\Rightarrow \psi$.
$\texttt{Prem}(A) = \texttt{Prem}(A_1) \cup \ldots \cup \texttt{Prem}(A_n)$, $\texttt{Conc}(A) = \psi$, $\texttt{Sub}(A) = \texttt{Sub}(A_1) \cup \ldots \cup \texttt{Sub}(A_n) \cup \{A\}$; $\texttt{TopRule}(A) = \texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow/\Rightarrow \psi$.

Each of these functions $\texttt{Func}$ are also defined on sets of arguments $S = \{A_1, \ldots, A_n\}$ as follows: $\texttt{Func}(S) = \texttt{Func}(A_1) \cup \ldots \cup \texttt{Func}(A_n)$. If an argument only uses strict rules, the argument is said to be *strict*. Otherwise it is *defeasible*. If an argument only has necessary premises, then the argument is *firm*, otherwise it is *plausible*. An argument $A_1, \ldots, A_n \rightarrow \varphi$ is called a *strict argument over* $\{\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n)\}$ and it is a *strict extension of* $A_1, \ldots, A_n$. A *basic defeasible argument* is an argument, which either has a defeasible top rule or is just an ordinary premise. The set of all basic defeasible arguments of a set of arguments S is denoted with $BD(S)$. The set of all arguments that just consist of a necessary premise are denoted with $NP(S)$.

Arguments can be attacked in three ways: on their premises (undermining attack), on their conclusion (rebutting attack) or on an inference step (undercutting attack). The latter two are only possible on applications of defeasible inference rules.

Definition 5. [**Attack**] A *attacks* B iff A *undercuts*, *rebuts* or *undermines* B, where:
• A *undercuts* argument B (on B') iff $\texttt{Conc}(A) = -n(r)$ and $B' \in \texttt{Sub}(B)$ such that B''s top rule r is defeasible.
• A *rebuts* argument B (on B') iff $\texttt{Conc}(A) = -\varphi$ for some $B' \in \texttt{Sub}(B)$ of the form $B''_1, \ldots, B''_n \Rightarrow \varphi$.
• Argument A *undermines* B (on B') iff $\texttt{Conc}(A) = -\varphi$ for some $B' \in \texttt{Sub}(B)$ of the form φ, $\varphi \in \mathcal{K}_p$.

Argumentation systems and knowledge bases together form argumentation theories.

Definition 6. [**Structured Argumentation Frameworks**] Let AT be an *argumentation theory* (AS, KB). A *structured argumentation framework* (SAF) defined by AT, is a triple $\langle \mathcal{A}, \mathcal{C}, \preceq \rangle$ where $\mathcal{A}$ is the set of all finite arguments constructed from KB in AS, $\preceq$ is an ordering on $\mathcal{A}$, and $(X, Y) \in \mathcal{C}$ iff X attacks Y.

The notion of a *defeat* can then be defined as follows. Undercutting attacks succeed as *defeats* independently of preferences over arguments, since they express exceptions to defeasible inference rules. Rebutting and undermining attacks succeed only if the attacked argument is not stronger than the attacking argument ($A \prec B$ is defined as usual as $A \preceq B$ and $B \not\preceq A$).

Definition 7. [**Defeat**] A *defeats* B iff A undercuts B or A rebuts/undermines B on B' and $A \not\prec B'$.

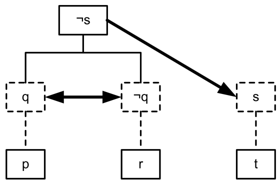

Figure 1. Arguments and attack relations

Abstract argumentation frameworks are then generated from $SAFs$ as follows:

Definition 8. [**Argumentation frameworks**] An *abstract argumentation framework (AF) corresponding to a* SAF = $\langle \mathcal{A}, \mathcal{C}, \preceq \rangle$ is a pair $(\mathcal{A}, D)$ such that D is the defeat relation on $\mathcal{A}$ determined by SAF.

2.1. The Problem Illustrated

The following abstract example illustrates the problems that can arise if the strict rules are derived from classical logic.

Example 9. Let $\mathcal{R}_d = \{p \Rightarrow q;\ r \Rightarrow \neg q;\ t \Rightarrow s\}$, $\mathcal{K}_p = \emptyset$ and $\mathcal{K}_n = \{p, r, t\}$, while $\mathcal{R}_s$ consists of all propositionally valid inferences. Then the corresponding AF includes the following arguments:
A_1: p $\quad A_2$: $A_1 \Rightarrow q$
B_1: r $\quad B_2$: $B_1 \Rightarrow \neg q$ $\quad C$: $A_2, B_2 \rightarrow \neg s$
D_1: t $\quad D_2$: $D_1 \Rightarrow s$
See also Figure 1, which displays the arguments and the attack relations. Dotted lines indicate defeasible inferences and dotted boxes indicate rebuttable conclusions.

Argument C attacks D_2. Whether C defeats D_2 depends on the argument ordering but plausible argument orderings are possible in which $C \not\prec D_2$ and so C defeats D_2. This is problematic, since s can be any formula, so any defeasible argument unrelated to A_2 or B_2, such as D_2, can, depending on the argument ordering, be defeated by C. Clearly, this is extremely harmful, since the existence of just a single case of mutual rebutting attack, which is very common, could trivialise the system.

In fact, the only semantics defined by Dung [14] that has problems with this example is grounded semantics. Since A_2 and B_2 attack each other and at least one of these attacks will (with non-circular argument orderings) succeed as defeat, all preferred or stable extensions contain either A_2 or B_2 but not both. And since both A_2 and B_2 attack C (by directly attacking one of its subarguments), C is for each preferred or stable extension defeated by at least one argument in the extension, so C is not in any of these extensions so D_2 is in all these extensions. However, if both A_2 and B_2 defeat each other, then neither of them is in the grounded extension, so that extension does not defend D_2 against C and therefore does not contain D_2.

Pollock [1,2] thought that this line of reasoning sufficed to show that his recursive-labelling approach (which was later by [15] proved to be equivalent to preferred semantics) adequately deals with this problem. However, Caminada [4] showed that the example can be extended in ways that also cause problems for preferred and stable semantics. Essentially, he replaced the facts p and r with defeasible arguments for p and r and let both these arguments be defeated by a self-defeating argument. Since such self-defeating arguments (if not themselves defeated) cannot be labelled *in* or *out*, their targets can also not be labelled *in* or *out*. But then neither A_2 nor B_2 can be labelled *in* or *out*, since they are both indirectly defeated by the self-defeating argument. Therefore neither A_2 nor B_2 will be in any preferred extension, so D_2 cannot be defended against C.

3. Deriving *ASPIC*$^+$'s Strict Rules from the Paraconsistent Logic W

In order to prevent the system from trivialising a natural approach is to replace classical logic as the source for strict rules with a logic that invalidates the Ex Falso principle. A logical consequence relation $\models$ is said to be paraconsistent if it is not 'explosive', i.e. when $\{a, \neg a\} \models b$ does not hold for any a and b. The idea of *ASPIC*$^+$ is that strict rules capture monotonic reasoning since attacks are not allowed on strictly derived conclusions. Therefore, the logic used for instantiating the strict rules should be monotonic. In this paper we explore the instantiation of *ASPIC*$^+$ with a so-called weak consequence relation originally proposed by [9], which is monotonic.

Definition 10. [**Weak consequence relation, logic** W] $\Gamma \vdash_W \alpha$ if and only if there is a maximal consistent subset Δ of Γ such that $\Delta \vdash \alpha$ in classical logic.

(Note that the maximality condition is redundant because of Lindenbaum's lemma, which states that every consistent set can be extended to a maximally consistent set.) It is easy to see that $\{a, \neg a\} \vdash_W b$ does not hold because $\{a, \neg a\}$ is not a maximal consistent subset of $\{a, \neg a\}$. Therefore, this consequence relation is paraconsistent. Below, the (non-)satisfaction of three important properties are described.

[R] $\alpha \in \Gamma$, then $\Gamma \vdash_W \alpha$.

Each $\alpha \in \Gamma$ belongs to some maximally consistent subset Δ of Γ. In classical logic, it holds that if $\alpha \in \Delta$, then $\Delta \vdash \alpha$. Therefore, it obviously holds that $\Gamma \vdash_W \alpha$.

[Mon] $\Gamma \vdash_W \alpha$, then $\Gamma, \Pi \vdash_W \alpha$.

The monotonicity property can be proven as follows. There must be maximal consistent subset Δ of Γ such that $\Delta \vdash \alpha$. Since $\Delta \subseteq \Gamma \cup \Pi$, there must exist a maximal consistent extension of Δ in $\Gamma \cup \Pi$, Δ', such that $\Delta' \vdash \alpha$. Therefore, $\Gamma, \Pi \vdash_W \alpha$.

[Cut] $\Gamma, \alpha \vdash_W \beta$ and $\Gamma \vdash_W \alpha$, then $\Gamma \vdash_W \beta$.

This rule does not hold, which is shown by a counterexample. Take the set $\Gamma = \{a, \neg a \wedge b\}$. Then $\Gamma \vdash_W b$ and $\Gamma, b \vdash_W a \wedge b$, while it is not the case that $\Gamma \vdash_W a \wedge b$.

Since the **Cut** rule does not hold, a naive instantiation of *ASPIC*$^+$'s strict rules with this logic W would lead to explosion:

Example 11. Take the following knowledge base $\mathcal{K}_p = \{p, \neg p, r\}$, $\mathcal{K}_n = \emptyset$, instantiate the strict rules with all valid inferences in the logic W and take $\mathcal{R}_d = \emptyset$. Then the following arguments can be constructed:

$A_1 : p$	$A_2 : A_1 \to p \vee \neg r$	$B : \neg p$
$C : A_2, B \to \neg r$	$D : r$	

Argument C concludes with $\neg r$ while $\mathcal{K}_p \nvdash_W \neg r$. The underlying reason for this problem is is that the **Cut** rule does not hold for $\vdash_W$. So if we want *ASPIC*$^+$'s strict part to behave according to $\vdash_W$, chaining of strict rules should be excluded.[1] To this end, we now redefine *ASPIC*$^+$'s notion of an argument:

[1] A similar idea was suggested to us by Martin Caminada in personal communication.

Definition 12. [**Argument* in *ASPIC****] An *argument** A on the basis of a knowledge base $\mathcal{K}$ in an argumentation system $(\mathcal{L}, \mathcal{R}, n)$ is:

1. φ if $\varphi \in \mathcal{K}$ with: $\texttt{Prem}(A) = \{\varphi\}$; $\texttt{Conc}(A) = \varphi$; $\texttt{Sub}(A) = \{\varphi\}$; $\texttt{TopRule}(A)$ = undefined.
2. $A_1, \ldots A_n \rightarrow \psi$ if $A_1, \ldots, A_n$ are arguments* with a defeasible top rule or are from $\mathcal{K}$ and $\mathcal{R}_s$ contains the rule $\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow \psi$.
3. $A_1, \ldots A_n \Rightarrow \psi$ if $A_1, \ldots, A_n$ are arguments* and R_d contains the rule $\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \Rightarrow \psi$.

For point 2 and 3: $\texttt{Prem}(A) = \texttt{Prem}(A_1) \cup \ldots \cup \texttt{Prem}(A_n)$, $\texttt{Conc}(A) = \psi$, $\texttt{Sub}(A) = \texttt{Sub}(A_1) \cup \ldots \cup \texttt{Sub}(A_n) \cup \{A\}$; $\texttt{TopRule}(A)$ = $\texttt{Conc}(A_1), \ldots, \texttt{Conc}(A_n) \rightarrow / \Rightarrow \psi$.

The rest of the framework remains unchanged. Moreover, from now on we assume that $\mathcal{L}$ is a propositional language and that $S \rightarrow \varphi \in \mathcal{R}_s$ iff S is finite and $S \vdash_W \varphi$.

4. Satisfaction of the Rationality Postulates

We next investigate whether the thus defined version of *ASPIC*$^+$ satisfies the consistency and closure postulates of [3]. Let E be any extension in the sense of [14] of an argumentation framework in the sense of Definition 8. Then the postulates are:

Closure under Strict Rules: $\texttt{Conc}(E) = Cl(\texttt{Conc}(E))$.
Direct Consistency: $\texttt{Conc}(E)$ is directly consistent.
Indirect Consistency: $Cl(\texttt{Conc}(E))$ is directly consistent.

Caminada, Amgoud, Modgil and Prakken [3,7,8] identify various sets of conditions under which all these postulates are satisfied. Briefly, the postulates are satisfied if first, the necessary premises are indirectly consistent and second, the strict rules are closed under either transposition or contraposition and third, the argument ordering is 'reasonable' in a certain sense. These results are useless for our purposes since with $\vdash_W$ as source of the strict rules, the system does not satisfy closure under transposition or contraposition. In fact, [13] show that any monotonic consequence notion that satisfies closure under transposition or contraposition will also satisfy explosion, while $\vdash_W$ as shown above does not satisfy explosion. We therefore adapt recent results of [13], who identify weaker conditions for satisfaction of the postulates. The core of these conditions is that any inconsistent set of formulas strictly implies the negations of all its members.

4.1. Logic-associated Argumentation Frameworks

Inspired by [16], [13] combine abstract argumentation with Tarskian abstract logics and investigate among other things an instantiation with *ASPIC*$^+$. Dung & Thang define a slightly generalised version of abstract logics as follows:

Definition 13. [**Abstract logic**] Given a language $\mathcal{L}$, an *abstract logic* is defined as a pair $(CN, CONTRA)$, where:

- $CN : 2^{\mathcal{L}} \longrightarrow 2^{\mathcal{L}}$ is a Tarski-like consequence operator over $\mathcal{L}$;
- $CONTRA \subseteq 2^{\mathcal{L}}$;

- for all sets of formulas $X, Y \subseteq \mathcal{L}$ it holds that:
  - $X \subseteq CN(X)$;
  - $CN(X) = CN(CN(X))$;
  - $CN(X) = \bigcup\{CN(Y) | Y \subseteq X$ and Y is finite$\}$;
  - if $X \in CONTRA$ then each superset of X also belongs to $CONTRA$. A set belonging to $CONTRA$ is said to be *contradictory*;
  - $CN(\emptyset)$ is not contradictory, i.e., $CN(\emptyset) \notin CONTRA$.

A set $X \subseteq \mathcal{L}$ is *inconsistent* if its closure $CN(X)$ is contradictory, otherwise X is *consistent*. X is closed if it coincides with its own closure $CN(X)$.

Definition 14. [**Logic-associated argumentation framework**] A *logic-associated argumentation framework* (LAF) over a language $\mathcal{L}$ is $(AF, \sqsubseteq, AL, Cnl)$, where:

- AF is an abstract argumentation framework.
- $AL = (CN, CONTRA)$ is an abstract logic over $\mathcal{L}$.
- $Cnl : \mathcal{A} \longrightarrow \mathcal{L}$ assigns to each argument A, its conclusion $Cnl(A)$ in $\mathcal{L}$.
- $\sqsubseteq$ is a partial order over $\mathcal{A}$ where $A \sqsubseteq B$ means that A is a subargument of B such that for all arguments $C \in \mathcal{A}$, if C attacks A then C attacks B.

4.2. Instantiating $LAFs$ with $ASPIC+$

Dung and Thang investigate an instantiation of LAFs with a version of *ASPIC*$^+$ with no preferences and a language consisting just of propositional literals. The AF is essentially defined as in Definition 8 above. They leave the knowledge base empty and represent facts F as strict rules $\rightarrow F$. It is easy to see that this is equivalent with instead putting F in $\mathcal{K}_n$. They then define $CONTRA_{AS}$ as the collection of all sets containing both p and $\neg p$ for some atom p. They define the consequence operator CN_{AS} in effect as $Cl_{\mathcal{R}_s}$ in *ASPIC*$^+$ (see Definition 3 above) and define the Cnl operator as *ASPIC*$^+$'s `Conc` operator.

We now redefine their instantiation of $LAFs$ with *ASPIC*$^+$for two reasons. First, we have changed *ASPIC*$^+$'s definition of an argument by forbidding chaining of strict rules. Second, because of this the notion of closure of a set of conclusions under strict rule application has to be changed, since we now want to exclude chaining of strict rules when considering such a closure. First we define for any set S of arguments*, $S^\#$ as the set of all basic defeasible arguments together with all necessary premises in S. Then the new definition of closure and indirect consistency are as follows.

Definition 15. [**Closure**] For any $X \subseteq \mathcal{L}$, let the closure of X under strict rules, denoted $Cl_{\mathcal{R}_s}(X)$, be the smallest set containing X and the consequent of any strict rule in $\mathcal{R}_s$ whose antecedents are in X.
The set of arguments* S is said to be *closed* iff $\texttt{Conc}(S) = Cl_{\mathcal{R}_s}(\texttt{Conc}(\texttt{Sub}(S)^\#))$.

Definition 16. [**Indirect consistency**] A set $X \subseteq \mathcal{L}$ is *indirectly inconsistent* if there is a $\varphi \in \mathcal{L}$ such that $\varphi, \neg\varphi \in Cl_{\mathcal{R}_s}(X)$. A set of arguments* S is said to be *inconsistent* (*consistent*) if $\texttt{Conc}(\texttt{Sub}(S)^\#)$ is indirectly inconsistent (resp. indirectly consistent).

We now investigate satisfaction of the strict-closure and indirect-consistency postulates as formulated with these redefined notions (direct consistency follows from indirect consistency). Below "consistency postulate" refers to the indirect-consistency postulate.

[13]'s instantiation with *ASPIC*$^+$ is without preferences and for languages consisting of literals. It turns out that their results can be easily generalised to the case with preferences included and for richer languages. Their proofs do not rely at all on the literal language so directly apply to any logical language. The same holds with preferences for their results on strict closure. Moreover, if the argument ordering is reasonable in the sense of [8] (see Definition 26 below), then their results also generalise to preferences. Because of space limitations we omit the proof of this result.

We can now now use [13]'s results as a guidance for proving the closure and consistency postulate for the logic W. At first sight, reusing [13]'s results would not seem to be possible since $\vdash_W$ does not satisfy idempotence (as shown by the counterexample to the **Cut** rule) and is therefore not an abstract logic. However, upon closer inspection it turns out that [13]'s results on *ASPIC*$^+$ do not use this property at all. It is then left to verify that their results still hold for the adapted notion of closure and its related adapted notion of (indirect) consistency.

We now adapt when necessary the definitions of [13] to the new notions of closure and consistency and then verify their results for the adapted definitions.

Definition 17. [**Argument* base**] A set of arguments* BA is a *base of an argument** A if the conclusion of A is a consequence of the conclusions of BA (i.e., if $\texttt{Conc}(A) \in Cl_{\mathcal{R}_s}(\texttt{Conc}(BA))$) and if each defeater[2] of A is a defeater of BA and vice versa.

In the example of Figure 1, the set $\{A_2, B_2\}$ is a base of C.

Definition 18. [**Generated arguments***]

- An argument* A is *generated* by a set of arguments* S if there is a base BA of A such that $BA \subseteq \texttt{Sub}(S)$.
- The set of all arguments* *generated* by S is denoted by $GN(S)$.

Definition 19. [**Compact**] An argumentation framework (AF) is *compact* if for each set of arguments* S, $GN(S)$ is closed. This is equal to $\texttt{Conc}(GN(S)) = Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S)^{\#})$.

Theorem 20. Let E be a complete extension, then $GN(E) = E$.

Proof. First note that according to Definition 18 for each set S of arguments* $\texttt{Sub}(S) \subseteq GN(S)$. Therefore $E \subseteq GN(E)$.
Suppose now that an argument* C defeats an argument* $A \in GN(E)$. Let BA be a base of A such that $BA \subseteq \texttt{Sub}(E)$, then C defeats BA. Hence C defeats $\texttt{Sub}(E)$ and so it defeats E. Since E is a complete extension, every defeat against E is counter defeated by E. A is defended by E, so $A \in E$. Therefore $GN(E) \subseteq E$. □

Theorem 21. Each compact AF satisfies the closure postulate.

Proof. Let E be a complete extension. The compactness implies that $GN(E)$ is closed. From Theorem 20 E is closed. □

Theorem 22. If the strict rules of an *ASPIC** AF are instantiated with all valid inferences of the logic W, then AF is compact.

[2]Note that Dung and Thang's notion of attack is renamed as defeat.

Proof. [$\texttt{Conc}(GN(S)) \supseteq Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$]
Let S be a set of arguments* and $\sigma \in Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$. It needs to be shown that $\sigma \in \texttt{Conc}(GN(S))$. Let X be a minimal subset of $\texttt{Conc}(\texttt{Sub}(GN(S))^{\#})$ such that $\sigma \in Cl_{\mathcal{R}_s}(X)$. Hence there is a strict argument* A_0 over X with conclusion σ. Further let S_X be a minimal set of arguments* from $\texttt{Sub}(GN(S))^{\#}$ s.t. $\texttt{Conc}(S_X) = X$. Let A be the argument* obtained by replacing each leaf in A_0 (viewed as a proof tree) labelled by a literal α from X by an argument* with conclusion α from S_X. Note that this is possible since all arguments* in S_X are basic defeasible arguments or are just necessary premises. It is obvious that the conclusion of A is σ. We show that S_X is a base of A. Suppose B is an argument* defeating A. Since A_0 is a strict argument* over X, B must defeat a basic defeasible subargument in S_X. Hence B defeats S_X. Thus $A \in GN(S)$. Hence $\sigma \in \texttt{Conc}(GN(S))$.
[$\texttt{Conc}(GN(S)) \subseteq Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$]
Suppose $\sigma \in \texttt{Conc}(GN(S))$, then it has to be shown that $\sigma \in Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$. $\sigma \in \texttt{Conc}(GN(S))$ means that there is an argument* $A \in GN(S)$ with $\texttt{Conc}(A) = \sigma$.
Suppose A is of the form $\Rightarrow \sigma$ or $\sigma \in \mathcal{K}_p$, then $A \in BD(\texttt{Sub}(GN(S)))$ and thus $A \in \texttt{Sub}(GN(S))^{\#}$.
Suppose A is of the form $\rightarrow \sigma$ or $\sigma \in \mathcal{K}_n$, then $\sigma \in Cl_{R_s}(\emptyset)$ or $\sigma \in NP(GN(S))$ respectively, so $\sigma \in Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$.
Suppose A is of the form $A_1, \ldots, A_n \Rightarrow \sigma$, then $A \in BD(\texttt{Sub}(GN(S)))$ and thus $A \in \texttt{Sub}(GN(S))^{\#}$.
Finally, suppose A is of the form $A_1, \ldots, A_n \rightarrow \sigma$, then since $A_1, \ldots, A_n$ are basic defeasible arguments $A_1, \ldots, A_n \in \texttt{Sub}(GN(S))^{\#}$. It can be concluded that $\sigma \in Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$.
Therefore $\sigma \in Cl_{\mathcal{R}_s}(\texttt{Sub}(GN(S))^{\#})$, so we have proved that AF is compact. □

Definition 23. **[Cohesive]** An AF is *cohesive* if for each inconsistent set of arguments* S, $GN(S)$ is conflicting.

Theorem 24. Each cohesive AF satisfies the consistency postulate.

Proof. Let E be a complete extension. Suppose E is inconsistent. From cohesion, it follows that $GN(E)$ is conflicting. Theorem 20 states that then E must be conflicting. This is a contradiction so E is consistent. □

Definition 25. **[Self-contradiction axiom]** An AF satisfies the *self-contradiction axiom* if for each minimal inconsistent set $X \in \mathcal{L}$ it holds that for all $\sigma \in X$, $\neg\sigma \in Cl_{\mathcal{R}_s}(X)$.

Definition 26. **[Reasonable argument ordering, [8]]** $\preceq$ is a *reasonable argument ordering* if and only if:

- $\forall A, B$, if A is strict and firm and B is plausible of defeasible, then $B \prec A$;
- $\forall A, B$, if B is strict and firm, then $B \not\prec A$;
- $\forall A, A', B, C$, such that $C \prec A$, $A \prec B$ and A' is a strict extension of $\{A\}$, then $A' \prec B, C \prec A'$;
- Let $\{C_1, \ldots, C_n\}$ be a finite subset of $\mathcal{A}$ and for $i = 1, \ldots, n$ let $C^{+/i}$ be some strict extension of $\{C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_n\}$. Then it is not the case that $\forall i,\ C^{+/i} \prec C_i$.

Theorem 27. If a compact AF has a consistent $\mathcal{K}_n$ and a reasonable argument ordering and satisfies the self-contradiction axiom, then AF is cohesive.

Proof. Let S be an inconsistent set of arguments* and take a minimal inconsistent subset S' of $\texttt{Sub}(S)$. Definition 16 combined with consistency of $\mathcal{K}_n$ and its minimality causes that $S' \neq \emptyset$ only contains basic defeasible arguments or necessary premises. Remark that S' cannot consist of only necessary premises because of consistency of $\mathcal{K}_n$. Since AF satisfies the self-contradiction axiom, for all $\sigma \in \texttt{Conc}(S')$ it holds that $\neg\sigma \in Cl_{\mathcal{R}_s}(\texttt{Conc}(S'))$. Let B be the weakest argument of S' with $\texttt{Conc}(B) = \sigma$. Note that B cannot be an ordinary premise because of the reasonable argument ordering and the fact that S' must contain basic defeasible arguments. From compactness combined with the fact that $\neg\sigma \in Cl_{\mathcal{R}_s}(\texttt{Conc}(S'))$ it follows that $\neg\sigma \in \texttt{Conc}(GN(S'))$. Therefore, there is an argument* $A \in GN(S')$ such that $\texttt{Conc}(A) = \neg\sigma$. Hence A attacks B. The base of A is S', so if A is defeated by some argument then S' must be under defeat. Therefore, it can be concluded that all basic defeasible subarguments of A are in S'. The fact that B is the weakest argument of S' in combination with the reasonable argument ordering implies that $A \not\prec B$. This means that A defeats B. Since $B \in S' \subseteq GN(S') \subseteq GN(S)$, $GN(S)$ is conflicting. Therefore, AF is cohesive. $\square$

Theorem 28. If the strict rules of an *ASPIC** AF are instantiated with all valid inferences of the logic W, then the AF satisfies the self-contradiction axiom.

Proof. It has to be proved that for every minimal inconsistent set $X \subseteq \mathcal{L}$ it holds that for each $\sigma \in X$, $\neg\sigma \in Cl_{\mathcal{R}_s}(X)$. Let X be a minimally inconsistent set and take $S = X \backslash \sigma$. Note that S is a maximal consistent subset of X and that $S, \sigma \vdash \bot$ (where $\vdash$ denotes classical entailment). By the deduction theorem for classical logic $S \vdash \sigma \supset \bot$, which implies $S \vdash \neg\sigma$. Since S is a maximal consistent subset of X, $X \vdash_W \neg\sigma$. This holds for every $\sigma \in X$, so $\neg\sigma \in Cl_{\mathcal{R}_s}(X)$. It can be concluded that AF satisfies the self-contradiction axiom. $\square$

Combining Theorem 21, 22, 24, 27 and 28 gives the following important conclusion.

Theorem 29. If the strict rules of an *ASPIC** AF are instantiated with all valid inferences of the logic W and if AF has consistent $\mathcal{K}_n$ and a reasonable argument ordering, then AF satisfies the closure and consistency postulate.

5. Conclusion

In this paper a long standing problem in the logical study of combined deductive and defeasible reasoning has been solved. The *ASPIC*$^+$ framework has been combined with the logic W [9] in a way that prevents trivialisation in case of rebutting arguments while preserving known results on consistency and strict closure. To obtain these results, the *ASPIC*$^+$ framework had to be adapted by prohibiting the chaining of strict rules, since the W logic does not satisfy the cut rule. In future research we intend to extend these results by investigating satisfaction of [5]'s postulates of non-interference and crash-resistance (like in [6]).

Since the logic W is monotonic, *ASPIC** as defined above allows for irrelevant arguments. For example, not only $p \rightarrow p \vee q$ is included in $\mathcal{R}_s$ but also $p, r \rightarrow p \vee q$ for any

r. In [17] versions of *ASPIC** (and also *ASPIC*$^+$) are investigated in which strict rules can only be applied to subset-minimal sets of formulas (clearly, this condition cannot be imposed on application of defeasible rules, since more specific defeasible rules are usually stronger than their less specific versions). It turns out that this minimality condition does not affect the outcomes on the natural assumption that arguments cannot be made weaker by deleting redundant subarguments from a strict-rule application. Similar investigations were made in the context of assumption-based argumentation by [18]. The combined results of the present paper and [17] shed light on the relation between *ASPIC** and classical argumentation as studied by [11], in which arguments are essentially classical proofs from consistent and subset-minimal subsets of a classical knowledge base: with the minimality condition on strict-rule application *ASPIC** is a proper extension of classical argumentation with defeasible rules and preferences.

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